SEQUENTIAL EQUILIBRIUM WITH CREDIT CONSTRAINTS

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ABSTRACT. The classic result by Magill and Quinzii (1996) for incomplete market economies with infinitely-lived assets shows that a competitive equilibrium may not exist when debt constraints or transversality conditions are used to prevent Ponzi schemes. By replacing the former with credit constraints targeting the amount of borrowing, we determine levels of liquidity under which a competitive equilibrium *always* exists. Our results include debt contracts with unbounded delivery streams and we do not require uniform impatience, although we must assume that preferences may be represented by time and state separable utility functions.

Keywords: General equilibrium, Incomplete markets, Credit constraints.

JEL classification: D52, D53, C62.

1. INTRODUCTION

The theory of general equilibrium with incomplete markets (GEI) provides a broader framework than the standard Arrow-Debreu model to study the conformation of both asset and commodity prices, and the interlinks between the later, investment and consumption (Geanakoplos 1990). Furthermore, infinite-horizon GEI models allow the study of several issues in financial markets, monetary and macroprudential policy, and consumption and investment dynamics of households and firms. Consequently, it is no surprise that the GEI model has become an integral tool for both theory and applied researchers in macroeconomics (Bayer et al. 2019; H. He and Pearson 1991) and finance (Chacko and Viceira 2005; Cochrane and Saa-Requejo 2000).

Nonetheless, there is a paucity of research studying conditions under which a competitive equilibrium in open-ended GEI models actually exists. Moreover, available results require stringent assumptions on agents' preferences, endowment streams and the structure and complexity of financial markets, limiting the extent to which GEI may be applied to study more applied contexts. This paper contributes to the literature by proving equilibrium existence in a whole new class of abstract economies and thus, takes a step towards closing the gap between the applied literature and available theoretical results. More precisely, we prove equilibrium existence in economies with long-lived debt contracts and where agents are subject to credit constraints limiting access to liquidity.

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Conventionally, the literature on sequential economies has imposed debt constraints or transversality conditions to prevent agents from incurring in the so-called Ponzi schemes, a step necessary to assert equilibrium existence. In a seminal paper, Magill and Quinzii (1994) study equilibrium existence in economies complying with uniform impatience, uniformly bounded endowments and short-lived securities paying in a numeraire commodity. Uniform impatience is a joint requirement on endowments and preference relations, which implies that agents' degree of impatience does not vanish asymptotically. In this context, they establish the equivalence between the competitive equilibria with an implicit debt constraint and the equilibria with a transversality condition.

However, when securities are real and long-lived, debt constraints and transversality conditions are insufficient to ensure that a competitive equilibrium *always* exists. Indeed, portfolios may fail to have endogenous Radner bounds when the rank of return matrices is price-dependent and agents may increase borrowing through investing more in financial markets. Thus, and even for finite horizon economies, a competitive equilibrium may cease to exist. As a consequence, available equilibrium results with multi-period securities are valid solely for dense subsets of economies (Hernández and Santos 1996; Magill and Quinzii 1996). Furthermore, there are no equilibrium results appliable to economies with infinitely lived debt contracts when delivery streams may be unbounded or when endowments are not uniformly bounded from below. Indeed, and as highlighted by Hernández and Santos (1996), in this setting uniform impatience is not enough to assert that finite asset prices are compatible with non-arbitrage, a necessary condition for equilibrium existence.

In this paper we follow an alternative approach and assume that agents are subject to credit constraints limiting borrowing to a proportion of the market value of aggregate wealth. These constraints were introduced by Moreno-García and Torres-Martínez (2012), who use them to assert equilibrium existence in economies whose financial markets are restricted to securities in positive net supply (equity contracts). More precisely, they show that when agents have time and state separable utility functions, credit constraints induce Radner bounds on portfolio holdings, solving the problem of non-existence of equilibrium that may arise under the presence of debt constraints or transversality conditions. In turn, we show that when liquidity is abundant enough, this result may be extended to economies with a general structure of securities in zero net supply. Therefore, and in the framework of Magill and Quinzii (1994, 1996), replacing transversality conditions with credit constraints leads to equilibrium existence *regardless* of the particularization of endowments or asset delivery streams.

The presence of credit constraints poses a technical challenge as it is no longer possible to jointly normalize commodity and security prices without compromising the continuity of budgetary correspondences. Thus, to assert equilibrium existence it is necessary to ensure that asset prices are endogenously bounded from above, which Moreno-García and Torres-Martínez (2012) accomplish for the particular case of promises in positive net supply. By ensuring that agents are solvent, in the sense that they may sell small amounts of available securities and honor their commitments with their future endowments, we show that credit constraints induce endogenous upper bounds on security prices when liquidity is abundant enough. In consequence, we are able to assert equilibrium existence under the additional requirement that the number of available securities is finite (see Theorem 2), result that we later extend to economies with an infinite number of securities by adapting the super-replication property by Cea-Echenique and Torres-Martínez (2016); see Theorem 3. Notably, these results hold even for economies in which endowments are not uniformly bounded or where asset deliveries conform unbounded sequences. Thus, we address the issue remarked by Hernández and Santos (1996) and provide equilibrium results for economies with infinitely lived debt contracts with unbounded delivery streams. Finally, we highlight that we do not require uniform impatience nor uniformly bounded endowment streams, but must assume instead that utility functions are time and state separable and unbounded.

The rest of the paper is organized as follows. Section 2 introduces our notation and the general model. Section 3 introduces a set of assumptions which are very similar to those present in Magill and Quinzii (1996), presents our first equilibrium result, and discusses why credit constraints are key to assert that an equilibrium exists with certainty. Section 4 extends our results to economies with unbounded endowment and asset delivery streams, describes the techniques we use to establish the existence of bounds on security prices, and compares our results with the problem posed by Hernández and Santos (1996). Section 5 presents a brief conclusion. All the proofs are relegated to the Appendix.

2. Model

Let $\mathcal{E}$ represent a discrete time, infinite horizon economy. There is a set $\mathcal{S}$ of states of nature characterizing uncertainty, which is homogeneous among agents and represented by a finite partition $\mathcal{F}_t$ of $\mathcal{S}$ at each period t. There is no information available at $t = 0$, i.e., $\mathcal{F}_0 = \mathcal{S}$. Additionally, $\mathcal{F}_t$ is at least as fine as $\mathcal{F}_{t-1}$ at every period $t \geq 0$. Thus, there is no loss of information throughout time.

A *node* ξ is characterized by a pair (t, σ), where $t \in \mathbb{N}$ and $\sigma \in \mathcal{F}_t$. Accordingly, $t(\xi)$ and $\sigma(\xi)$ denote the date and the information set associated to ξ. Let ξ^-, and ξ^+ be, respectively, node ξ's unique predecessor and the (finite) set of all its immediate successors. We also say $\mu \geq \xi$ whenever $t(\mu) \geq t(\xi)$ and $\sigma(\mu) \subseteq \sigma(\xi)$, so $\xi^+ = \{\mu \geq \xi | t(\mu) = t(\xi) + 1\}$. Analogously, $\mu > \xi$ indicates that $\mu \geq \xi$ and $\mu \neq \xi$.

There is a unique initial node ξ_0, marking the beginning of the *event-tree* $\mathcal{D}$, formed by the set of nodes in our economy. The *subtree* starting at node ξ is denoted by $\mathcal{D}(\xi)$ and corresponds to the set $\{\mu \in \mathcal{D}\mu \geq \xi\}$. Moreover, define $\mathcal{D}^t(\xi) = \{\mu \in \mathcal{D}(\xi) | t(\mu) \leq t(\xi) + t\}$ as the branch of $\mathcal{D}(\xi)$ spanning until date t. Similarly, $\mathcal{D}_t(\xi) = \{\mu \in \mathcal{D}(\xi) : t(\mu) = t + t(\xi)\}$ is the set of nodes in $\mathcal{D}(\xi)$ whose dates coincide with $t + t(\xi)$.

Markets. There is a finite and ordered set $\mathcal{L}$ of perfectly divisible and perishable commodities, available at every $\xi \in \mathcal{D}$. Therefore, the set consisting of all commodities indexed across the event tree is $\mathcal{D} \times \mathcal{L}$. We denote as $p(\xi) = (p_l(\xi))_{l \in \mathcal{L}}$ the vector of commodity spot prices at ξ, and $p = (p(\xi))_{\xi \in \mathcal{D}}$ as the commodity price process along $\mathcal{D}$.

There is a ordered set $\mathcal{J}$ of financial assets in zero net supply. Every $j \in \mathcal{J}$ is characterized by an issuing node $\xi_j \in \mathcal{D}$ and a payment stream consisting of a non-trivial process of commodity bundles $A_j = (A_j(\mu))_{\mu > \xi_j} \in \mathbb{R}^{\mathcal{L} \times \mathcal{D}(\xi_j) \setminus \xi_j}$. Securities in $\mathcal{E}$ may be finitely or infinitely lived. Precisely, asset $j \in \mathcal{J}$ is finitely lived if there exists $T_j \in \mathbb{N}$ such that the set $\{\xi \in \mathcal{D} | (t(\xi) > T_j) \wedge (A_j(\xi) > 0)\}$ is empty. In turn, a security $j \in \mathcal{J}$ is infinitely lived if the latter does not hold for any $T \in \mathbb{N}$. Let $A = (A_j)_{j \in \mathcal{J}}$ denote the security payoff process.

At ξ, there is a finite set $\mathcal{J}(\xi)$ of securities available for trade; asset spot prices at ξ are denoted as $q(\xi) = (q_j(\xi))_{j \in \mathcal{J}(\xi)}$. Let $\mathcal{D}(\mathcal{J}) = \{(\xi, j) \in \mathcal{D} \times \mathcal{J} : j \in \mathcal{J}(\xi)\}$, and let the space of commodity and asset prices be $\mathbb{P} := \mathbb{R}_+^{\mathcal{D} \times \mathcal{L}} \times \mathbb{R}_+^{\mathcal{D}(\mathcal{J})}$. Accordingly, let $\mathcal{A} := \mathbb{R}_+^{\mathcal{D}(\mathcal{J}) \times \mathcal{L}}$ be the security payoff process's space.

Agents. There is a finite set $\mathcal{H}$ of agents participating in the economy, each one of them characterized by an utility function $U^h : \mathbb{R}_+^{\mathcal{D} \times \mathcal{L}} \to \mathbb{R}_+ \cup \{+\infty\}$, and commodity endowments $(w^h(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}_+^{\mathcal{D} \times \mathcal{L}}$. Aggregate wealth at node ξ is therefore $W(\xi) = \sum_{h \in \mathcal{H}} w^h(\xi)$; let $W = (W(\xi))_{\xi \in \mathcal{D}}$.

Each $h \in \mathcal{H}$ must choose an allocation $(x^h(\xi), \theta^h(\xi), \varphi^h(\xi)) \in \mathbb{E}_\xi := \mathbb{R}^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}(\xi)} \times \mathbb{R}_+^{\mathcal{J}(\xi)}$ for every $\xi \in \mathcal{D}$, composed by commodity bundles and long and short positions in financial securities. Accordingly, $(x^h(\xi), \theta^h(\xi), \varphi^h(\xi))_{\xi \in \mathcal{D}} \in \mathbb{E} := \prod_{\xi \in \mathcal{D}} \mathbb{E}_\xi$. Given prices $(p, q) \in \mathbb{P}$, an allocation $(x, \theta, \varphi) \in \mathbb{E}$ is said to be *budget feasible* for agent $h \in \mathcal{H}$ if it complains with the following restriction at every node:

$$p(\xi)\left(x^h(\xi) - w^h(\xi)\right) + q(\xi)\left(\theta^h(\xi) - \varphi^h(\xi)\right) \leq \sum_{j \in \mathcal{J}(\xi^-)} \left(p(\xi) A_j(\xi) + q_j(\xi)\right)\left(\theta_j^h(\xi^-) - \varphi_j^h(\xi^-)\right).$$

We use $y^h(\xi) = (x^h(\xi), \theta^h(\xi), \varphi^h(\xi))$, and $y^h = (y(\xi))_{\xi \in \mathcal{D}}$, to shorten notation.

Furthermore, we assume that every agent h is subject to a series of *credit constraints* linking short-sales at node ξ to the overall availability of resources in the economy. Thus, she is restricted to choose allocations belonging to the following choice set:

$$\mathcal{C}^h(p, q, \kappa) = \left\{ y \in \mathbb{E} \;\middle|\; \begin{array}{c} y \text{ is budget feasible for } h \\ q(\xi)\varphi(\xi) \leq \kappa p(\xi) W(\xi), \ \forall \xi \in \mathcal{D} \end{array} \right\}. \tag{1}$$

Parameter $\kappa > 0$ is exogenously determined, and may be understood as constraining liquidity in the economy, as borrowing at node ξ is limited to a proportion κ of the market value of aggregate wealth.

We are now ready to introduce our definition of a competitive equilibrium for economy $\mathcal{E}$.

Definition 1. A *competitive equilibrium* with credit constraints for economy $\mathcal{E}$ is composed by a pair of price processes $(p, q) \in \mathbb{P}$ and a set of allocations $(y^h)_{h \in H} \in \mathbb{E}^{\mathcal{H}}$ such that

(1) For every $h \in \mathcal{H}$, y^h is maximal regarding U^h in $\mathcal{C}^h(p, q, \kappa)$.

(2) Commodity and financial markets clear, i.e., at every $\xi \in \mathcal{D}$:

$$\sum_{h \in \mathcal{H}} x^h(\xi) = W(\xi) \text{ and } \sum_{h \in \mathcal{H}} \theta^h(\xi) = \sum_{h \in \mathcal{H}} \varphi^h(\xi).$$

Moreno-García and Torres-Martínez (2012) introduced choice set (1) and used it to prove equilibrium in an infinite horizon economy with long-lived real securities in positive net supply. Remarkably, under certain conditions the presence of credit constraints induces endogenous Radner bounds on portfolio holdings, which discards both Ponzi schemes and the indeterminacy of portfolio sizes that may appear in a similar setting with debt constraints or transversality conditions. In the coming sections, we will study conditions under which credit constraints may also be used to induce endogenous upper bounds on security prices, which in turn allows to assert equilibrium existence with long-lived debt contracts.

3. Equilibrium existence with credit constraints

We assume that agents' preferences may be represented by time and state separable utility functions, and that period utility functions are concave, strictly increasing and unbounded. Furthermore, we follow Magill and Quinzii (1994) and assume that endowments are uniformly bounded from below and that aggregate wealth is uniformly bounded from above.

Assumption A1 (Utility)**.** *$U^h(x) = \sum_{\xi \in \mathcal{D}} u^h(\xi, x(\xi))$ for each $h \in \mathcal{H}$, where $u^h(\xi, .) : \mathbb{R}^{\mathcal{L}}_{+} \to \mathbb{R}_{+}$ is continuous, concave, strictly increasing and unbounded. Moreover, it holds that $U^h(0) = 0$.*

Assumption A2 (Endowments)**.** *There exist $\omega, \omega' \in \mathbb{R}^{\mathcal{L}}_{++}$ such that $\omega \leq w^h(\xi)$ and $W(\xi) \leq \omega'$ for every $(h, \xi) \in \mathcal{H} \times \mathcal{D}$. Also, it holds that $U^h(W) < +\infty$ for every $h \in \mathcal{H}$.*

Date-event separability of utility functions is common ground in the GEI literature (Araujo, Páscoa, and Torres-Martínez 2002, 2011; Moreno-García and Torres-Martínez 2012), and is useful to approximate the equilibrium of our infinite horizon economy through competitive equilibria in truncated, finite horizon economies. Furthermore, and jointly with our requirements on period utility functions, it allows to determine both lower and upper bounds on security prices. Unboundedness of utility functions $(u^h(\xi, .))_{\xi \in \mathcal{D}}$ implies that consumption is non-essential across nodes, as the utility loss due to a decrease in consumption at node ξ may be compensated by a sufficiently large increase in consumption at any $\mu \neq \xi$. This property is not a requirement in Magill and Quinzii (1994, 1996) or Hernández and Santos (1996) who assume instead uniform impatience. Uniform impatience is a joint assumption regarding endowments and preference relations, that ensures that agents' degree of impatience does not vanish asymptotically. As shown by Bewley (1972), preferences complying with uniform impatience may be represented by utility functions of the sort:

$$U(x) = \sum_{\xi \in \mathcal{D}} \beta^{t(\xi)} \rho(\xi) v(\xi),$$

where $\beta \in (0, 1)$ is a discount factor, $\rho(\xi)$ is the probability of ξ, and $v : \mathbb{R}^{\mathcal{L}}_{+} \to \mathbb{R}$ is a continuous, increasing and concave satisfying $v(0) = 0$. To the extent that v may be bounded, we do not capture all possible utility functions consistent with uniform impatience. In turn, our setup does allow for utilities exhibiting non-recursive features such as hyperbolic discounting, which are ruled out under the former (Páscoa, Petrassi, and Torres-Martínez 2011).

The following assumptions involve the structure of financial markets. Similarly to Magill and Quinzii (1994), we impose bounded asset delivery streams, and require the availability of a risk-free console.

Assumption A3 (Delivery streams). *For every $j \in \mathcal{J}$, it holds that $A_j \in \ell^+_\infty(\mathcal{D} \times \mathcal{L})$, where $\ell^+_\infty(\mathcal{D} \times \mathcal{L})$ equals the subset of $\mathbb{R}^{\mathcal{D}\times\mathcal{L}}$ consisting of all non-negative, bounded sequences.*

Assumption A4 (Riskless asset). *Since ξ_0 there is available an infinitely lived security j' paying one unit of every commodity at every node:*

$$A_{j',l}(\xi) = 1 \quad \textit{for every } (l, \xi) \in \mathcal{L} \times \mathcal{D} \setminus \xi_0.$$

The presence of credit constraints as in (1) poses a methodological challenge as it is no longer possible to jointly normalize commodity and asset prices in the unitary simplex without compromising the continuity of budget sets.[1] Hence, to assert equilibrium existence it is necessary to make sure that asset prices are endogenously bounded from above. Moreno-García and Torres-Martínez (2012) proved this holds in the particular case of securities in positive net supply, but where unable to extend their results to a more general setting of financial promises, something we accomplish under the requirement that the value of κ is large enough. This leads to our first equilibrium result.

Theorem 1. *Under Assumptions A1-A4 there exists $\hat{\kappa} \in \mathbb{R}_{++}$ such that $\mathcal{E}$ has a competitive equilibrium whenever $\kappa \geq \hat{\kappa}$.*

Abundant liquidity, embodied by a sufficiently high value of κ, is a necessary condition for our equilibrium results. Indeed, when $\kappa \geq \hat{\kappa}$ agents are able to borrow enough resources as to buy commodity bundles that are inconsistent with the economy's availability of resources at key nodes, which in turns allows us to assert that security prices are bounded when agents are able to sell available financial securities. Although a more thorough description on the techniques used to bound security prices is provided in the coming section, it is relevant to remark that $\hat{\kappa}$ is determined as a function of our economy's fundamentals.

Remark 1. The threshold $\hat{\kappa}$ is determined by the utility attainable by consuming the totality of aggregate wealth, $(U^h(W))_{h\in\mathcal{H}}$, and does not depend on the arbitrary election of $A \in \mathcal{A}$ nor the particular redistribution of agents' endowment streams $(w^h)_{h\in\mathcal{H}}$ as long as W remains constant.

In combination with our assumptions, credit constraints induce endogenous Radner bounds on short-sale portfolios. Indeed, and because Assumptions A1-A2 ensure that asset prices are endogenously bounded away from zero, the short-sales of any security at ξ cannot exceed the quotient of the market value of aggregate wealth by that security's price lower bound at that node. In turn, and throughout

[1] Indeed, when commodity and security prices are normalized into the unitary simplex budget sets may have an empty interior when the former have zero weight relative to the latter. When this holds, standard techniques used to assert the lower hemicontinuity of budget sets fail to work; see Cea-Echenique and Torres-Martínez (2016) for a more detailed discussion.

an aggregation argument, this result allows to assert that portfolio sizes are endogenously bounded. Thus, credit constraints not only prevent agents from engagin into Ponzi schemes, but also solve the portfolio indeterminacy problem present when budget sets are defined in terms of debt constraints or transversality conditions. Therefore, and in contrast with the results of Magill and Quinzii (1996) and Hernández and Santos (1996), we are able to guarantee equilibrium existence for all economies complying with our assumptions, regardless of the particularization of endowments or the asset payoff process.

4. Generalization to unbounded endowment and asset delivery streams

In Section 3 we discussed how credit constraints, although quite useful to guarantee equilibrium existence, introduced the necessity of finding endogenous bounds on security prices. In this section we will discuss a series of requirements over the structure of financial markets which are instrumental to accomplish the former objective. By doing so, we will be able to substantially relax the requirements of the previous section, and provide equilibrium results for economies in which neither endowments are uniformly bounded nor aggregate wealth or asset delivery streams are bounded. Hernández and Santos (1996) showed that in this context portfolio size indeterminacy is not the only reason a competitive equilibrium may cease to exist. Indeed, when infinitely lived assets may have unbounded delivery streams, or endowments are not uniformly bounded from below, finite asset prices may be no longer compatible with non-arbitrage. Example 1 illustrates this point.

Example 1 (adapted from Hernández and Santos 1996, Example 3.9, p.118). Let $\mathcal{E}$ be an infinite horizon economy with no uncertainty and a unique perishable commodity. There is a representative consumer with an endowment stream $w = \{w_t\}_{t\geq 0}$, and with preferences over consumption that are represented by the utility functional $U(x) = \sum_{t\geq 0} \beta_t u(x_t)$, where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, derivable, strictly increasing and concave function, and $\beta_t \in \mathbb{R}_+$ for every $t \in \mathbb{N}$. Furthermore, we assume $U(0) = 0$, $U(w) < +\infty$ and $\sum_{t\geq 0} \beta_t < +\infty$.

There is only one asset available, issued at $t = 0$, which is infinitely lived and delivers a non-negative payment stream $A = \{A_t\}_{t>0}$. Moreover, the consumer is restricted to choose debt portfolios $(\varphi_t)_{t\geq 0}$ complying with $q_t\varphi_t \leq \kappa w_t$ at every t, with $\kappa > 0$.

$\mathcal{E}$ may not have equilibrium depending on the particularization of processes $\{w_t, \beta_t, A_t\}_{t\geq 0}$. Indeed, if an equilibrium for $\mathcal{E}$ were to exist, the first order conditions of the representative consumer's problem entail that the (finite) asset price at t must comply with:

$$q_t = \frac{1}{\beta_t u'(w_t)} \left[\sum_{s>t}^{T-1} (\beta_s u'(w_s) A_s) + \beta_T u'(w_T) q_T \right], \text{ for any } T > t. \tag{2}$$

The latter equation does not necessarily lead to a finite price: a constant endowment process, jointly with any pair of processes $\{\beta_t, A_t\}_{t\geq 1}$ such that $\sum_{t\geq 0} \beta_t A_t = +\infty$ leads to the opposite. In particular, consider $w_t = 1$ and $\{\beta_t, A_t\} = \{0.5^t, 2^t\}$ for all $t \geq 1$.[2] Note that uniform impatience

[2] Alternatively, a constant delivery stream with any pair of processes $\{\beta_t, w_t\}_{t\geq 1}$ such that $\sum_{t\geq 0} \beta_t u'(w_t) = +\infty$ also leads to non-existence.

holds for this particular case, as endowments are uniformly bounded and the utility function complies with the properties mentioned in Section 3; still, it is not possible to assert equilibrium existence.

In turn, to assert equilibrium existence it is enough to ensure that there exists a number $\delta \in \mathbb{R}_{++}$ such that $\delta A_t \leq w_t$ holds at every $t \geq 1$. Indeed, we may use the concavity of $u(.)$ plus the fact that utility is finite on aggregate wealth to check that price:

$$\bar{q}_t = \frac{1}{\beta_t u'(w_t)} \left[\sum_{s>t}^{+\infty} (\beta_s u'(w_s) A_s) \right],$$

is in fact an equilibrium price for the financial security at t. Intuitively, scaling asset deliveries in terms of aggregate wealth (which retrieves finite utility) implies that the former cannot be used to attain arbitrary levels of utility in equilibrium. This ensures the existence of finite prices compatible with (2).

From Example 1 we learn that it is divergence between deliveries and endowment streams which may lead to non-existence in representative agent economies. Notably, this intuition proves to be valuable for the more general case, as we are able to induce bounds on asset prices as long as their deliveries do not disproportionately outgrow the endowments of all agents in the economy. We now proceed to reinstate our assumptions as to present the aforementioned results.

Assumption B1 (Utility)**.** *$U^h(x) = \sum_{\xi \in \mathcal{D}} u^h(\xi, x(\xi))$ for each $h \in \mathcal{H}$, where $u^h(\xi, .) : \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}_+$ is continuous, concave, strictly increasing and unbounded. Moreover, it holds that $U^h(0) = 0$.*

Assumption B2 (Endowments)**.** *For every (ξ, h) in $\mathcal{D} \times \mathcal{H}$ it holds that $w^h(\xi) \gg 0$. Also, it holds that $U^h(W) < +\infty$ for every $h \in \mathcal{H}$.*

Assumption B1 is identical to its counterpart in Section 3; we repeat it for clarity of exposition. However, Assumption B2 implies a significant relaxation of Assumtion A2, as only requiring interior endowments implies that both individual and aggregate endowments may grow without bound. The following is a joint requirement on the delivery streams of a subset of available securities and endowment streams, anchoring the growth of the former to the one of the latter. Note that it is closely connected to the insights provided by Example 1.

Assumption B3 (Solvency)**.** *There exists a finite set $\mathcal{K} \subset \mathcal{J}$ such that for each $k \in \mathcal{K}$ there exists $\delta \in \mathbb{R}_{++}$ for which it holds that:*

$$\delta A_k(\mu) \leq w^h(\mu),$$

for some $h \in H$ and for every $\mu > \xi_k$.

Remark 2. Assumption B3 holds if:

(1) Securities are finitely lived and endowments are interior.

(2) Asset deliveries are bounded and endowments are uniformly bounded from below.

(3) Securities are in positive net supply (as their deliveries should be counted as part of agents cummulative endowments).

Thus, Assumption B3 holds in the models by Magill and Quinzii (1996), Hernández and Santos (1996) and Moreno-García and Torres-Martínez (2012).

Importantly, Assumption B3 implies that some agents are solvent, in the sense that they may sell small amounts of securities in $\mathcal{K}$ and honor their commitments with their future endowment streams. This solvency property, jointly with a sufficient level of liquidity, allows to assert that the prices of securities in $\mathcal{K}$ are endogenously bounded from above. In turn, this allows us to prove equilibrium existence under the additional requirement that $\mathcal{J}$ is restricted to $\mathcal{K}$.

Theorem 2. *Assume that $\mathcal{J} = \mathcal{K}$. Then, under Assumptions B1-B3 there exists $\hat{\kappa} \in \mathbb{R}_{++}$ such that $\mathcal{E}$ has a competitive equilibrium whenever $\kappa \geq \hat{\kappa}$.*

Importantly, it is not possible to extend this result to economies endowed with infinite securities, as the liquidity requirements for equilibrium existence may become indeterminate. Indeed, threshold $\hat{\kappa}$ is determined as a function of a series of bundles $(a(\xi))_{\xi \in D(\mathcal{K})}$ complying with

$$u^h(\xi, a(\xi)) > U^h(W) \quad \text{for all } h \in \mathcal{H},$$

and where $\mathcal{D}(\mathcal{K}) = \{\xi \in \mathcal{D} \mid \xi = \xi_k \text{ for some } k \in \mathcal{K}\}$. Although Assumptions B1 and B2 ensure that such a bundle exists for every node in $\mathcal{D}$, it is crucial for $\mathcal{D}(\mathcal{K})$ to have finite cardinality as to ensure that $\hat{\kappa}$ is correctly defined. Example 2 shows why this is the case.

Example 2. Consider an economy $\mathcal{E}$ with a single, perishable commodity and where agents share the same preferences over consumption streams represented by an utility function of the sort:

$$U^h(x) = \sum_{\xi \in \mathcal{D}} \beta^{t(\xi)} \rho(\xi) \sqrt{x(\xi)}, \quad \forall h \in \mathcal{H},$$

where $\beta \in (0,1)$ is a discount factor and $\rho(\xi)$ is the probability of realization of node ξ in $\mathcal{D}$. Moreover, assume that every agent in the economy shares the same constant endowment stream $w^h(\xi) = \omega$, for all $(\xi, h) \in \mathcal{D} \times \mathcal{H}$.

Under Assumption B2, it holds that $U^h(W) < +\infty$ for all agents; let $U(W) = \max_{h \in \mathcal{H}} U^h(W)$. To ensure that it provides an utility greater than $U(W)$ at ξ, bundle $a(\xi)$ must comply with:

$$a(\xi) > \left(\frac{U(W)}{\beta^{t(\xi)} \rho(\xi)} \right)^2.$$

Therefore, for agents to be able to purchase $a(\xi)$ through borrowing at ξ it must hold that:

$$\kappa \geq \hat{\kappa}_\xi = \frac{1}{\omega} \times \left(\frac{U(W)}{\beta^{t(\xi)} \rho(\xi)} \right)^2.$$

Clearly, liquidity requirements $\hat{\kappa}_\xi$ increase with $t(\xi)$, and thus, it is not possible to determine the existence of a uniform threshold $\hat{\kappa}$ whenever $\mathcal{J}$ contains an infinite number of securities appearing at a infinite number of different nodes.

To prove equilibrium existence in economies with infinite securities, we adapt the super-replication property introduced by Cea-Echenique and Torres-Martínez (2016) in the context of two-period economies with financial segmentation. The super-replication of a promise requires the existence of a portfolio yielding greater deliveries in every possible date-event; intuitively, the equilibrium price of a security should not exceed that of its super-replicating portfolio. Thus, by assuming that securities in $\mathcal{K}$ may be used to super-replicate the rest of the securities in $\mathcal{J}$, we are able to assert that the price of the latter is bounded in equilibrium.[3]

Assumption B4 (Super-replication). *For each asset $j \in \mathcal{J} \setminus \mathcal{K}$ there is a portfolio $\hat{\theta} = (\hat{\theta}_k)_{k\in\mathcal{K}}$ such that:*

$$A_j(\mu) \leq \sum_{k\in\mathcal{K}} A_k(\mu)\hat{\theta}_k \quad \textit{for all } \mu > \xi_j.$$

Remark 3. Assumption B4 does not render securities in $\mathcal{J} \setminus \mathcal{K}$ irrelevant in the sense that they may not be traded in equilibrium or have no effects over welfare. Indeed, super-replicated securities maintain their potential to complete markets or allow agents to achieve more efficient allocations of resources.

Theorem 3. *Under Assumptions A1-A4 there exists $\hat{\kappa} \in \mathbb{R}_{++}$ such that $\mathcal{E}$ has a competitive equilibrium whenever $\kappa \geq \hat{\kappa}$.*

Remark 4. Theorem 1 may be obtained as a corollary of Theorem 3. Indeed, because endowments are uniformly bounded from below Assumption B3 holds for the risk-free console, which may later be used to super-replicate the rest of available securities with bounded delivery streams.

Similarly to the results exposed in Section 3, Theorems 2 and 3 hold independently of the election of the endowment process $(w^h(\xi))_{h\in H}$ and the asset payment process $A \in \mathcal{A}$, as long as they comply with our assumptions. Hence, we remark that our results are not limited to dense subsets of parameters. Furthermore, we are the first to prove equilibrium existence with infinitely lived debt contracts in economies without uniformly bounded endowments or bounded delivery streams. Indeed, and in contrast with Hernández and Santos (1996) we are able to assert that finite security prices are always compatible with non-arbitrage. This holds partly because Assumptions B3 and B4 ensure that asset delivery streams remain proportional to aggregate wealth and thus, discard the divergence that lead to non-existence in Example 1.[4]

Remark 5. Parameter $\hat{\kappa}$ implements sufficiently high levels of liquidity as to allow agents to purchase bundles $(a(\xi))_{\xi\in\mathcal{D}(\mathcal{K})}$ through borrowing, an outcome that is never feasible in equilibrium. Nevertheless, we cannot assert that liquidity is too much, in the sense that credit constraints become

[3]Iraola, Sepúlveda, and Torres-Martínez (2018) also rely on super-replication to derive endogenous bounds on security prices in an infinite horizon economy with collateralized markets and financial segmentation.

[4]Hernández and Santos (1996) also require one out of two regularity conditions: that either there exists a self-financing portfolio super-replicating aggregate wealth (Assumption B4), or that each agent owns a proportion of aggregate wealth (Assumption B5). None of them need to hold under our assumptions.

non-binding. Indeed, we are unable to discard that, in equilibrium, some $h \in \mathcal{H}$ may be choosing an allocation $(x^h, \theta^h, \varphi^h) \in C^h(p, q, \kappa)$ such that $q(\xi)\varphi(\xi) = p(\xi)W(\xi)$ at any number of nodes $\xi \in \mathcal{D}$. This is in contrast with Magill and Quinzii (1994, 1996), who establish existence results for economies with non binding debt constraints.

Remark 6. Unequal access to financial markets may be relevant to understand the prevalence of a wide range of phenomena in financial markets, such as negative equity loans (Iraola and Torres-Martínez 2014), asset pricing puzzles (Guvenen 2009; Gromb and Vayanos 2017) and may even play an important role in determining the impact of macroprudential policies (Vayanos and Vila 2009; Chen, Cúrdia, and Ferrero 2012; Z. He and Krishnamurthy 2013). Our equilbrium results are consistent with broad form of credit segmentation. Indeed, and in a spirit similar to Cea-Echenique and Torres-Martínez (2016) and Faias and Torres-Martínez (2017), we may incorporate exogenous trading constraints limiting agents' participation in financial markets. Therefore, and depending on the nature of trading constraints, agents may be barred from trading certain securities, or have their access to credit markets conditioned on their previous allocations. Finally, we may have agents with no access to credit whatsoever.

5. Conclusion

In this paper, we present conditions under which a competitive equilibrium always exists in infinite horizon economies with long-lived real assets and incomplete markets. In particular, we show that when agents have time and state separable utility functions an are subject to credit constraints restricting borrowing, portfolio sizes are endogenously bounded independently of the election of security payoffs. Moreover, when liquidity is abundant enough, credit constraints may be used to induce endogenous upper bounds on security prices, a result that holds even when endowments are not uniformly bounded or when asset deliveries may conform unbounded sequences. Thus, our equilibrium results hold for economies with unrestricted economic growth and infinitely lived securities with unbounded payoff streams.

An issue left for future research is the study of the occurrence of rational bubbles in the prices of debt contracts under the presence of credit constraints. Indeed, and as highlighted by Magill and Quinzii (1996) and Moreno-García and Torres-Martínez (2012), in this setting rational bubbles may have real effects.

6. Appendix

We only provide a complete proof of Theorem 3, as both Theorem 1 and 2 may be obtained as direct corollaries of Theorem 3. Indeed, Theorem 1 is the particular case with uniformly bounded endowments and bounded asset delivery streams, and where the set $\mathcal{K}$ consists of the risk console described in Assumption A4. Theorem 2 is just the case in which $\mathcal{J} \setminus \mathcal{K} = \emptyset$.

Proof of Theorem 3

Truncated finite horizon economies. Let $\mathcal{E}_n^T$ be the finite horizon version of economy $\mathcal{E}$ up to time $T \in \mathbb{N}$ where, additionally, the net supply of financial securities has been increased by an amount proportional to $\frac{1}{n} > 0$, for $n \in \mathbb{N}$ given. In particular, $\mathcal{E}_n^T$ starts at node ξ_0 and is circumscribed to event-tree $\mathcal{D}^T(\xi_0)$. At every node $\xi \in \mathcal{D}^{T-1}(\xi_0)$ there is a set $\mathcal{J}^T(\xi) = \{j \in \mathcal{J} | \exists \mu > \xi : t(\mu) < T, A_j(\mu) \neq 0\}$ of assets available for trade; $\mathcal{J}^T(\xi) = \emptyset$ for all $\xi \in \mathcal{D}_T(\xi)$, by assumption. Let $\mathcal{K}^T(\xi) = \{k \in \mathcal{J}^T(\xi) : \ k \in \mathcal{K}\}$. Importantly, given $\xi \in \mathcal{D}^{T-1}(\xi_0)$, $\mathcal{J}^T(\xi) = \mathcal{J}(\xi)$ for T large enough. We will use $\mathcal{J}^T$ and $\mathcal{K}^T$ to denote the subset of securities of $\mathcal{J}$ and $\mathcal{K}$ which are available for trade at some node in $\mathcal{E}_n^T$. Finally, let $\mathcal{D}^T(\mathcal{J}) = \{(\xi, j) \in \mathcal{D}^T(\xi_0) \times \mathcal{J}^T(\xi) : j \in \mathcal{J}^T(\xi)\}$.

We consider prices (p, q) belonging to space

$$\mathbb{P}^T = \prod_{\xi \in \mathcal{D}^{T-1}(\xi_0)} \left(\Delta_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}^T(\xi)} \right) \times \prod_{\xi \in \mathcal{D}_T(\xi_0)} \Delta_+^{\mathcal{L}},$$

where $\Delta_+^{\mathcal{L}} := \{p \in \mathbb{R}_+^{\mathcal{L}} : \|p\|_\Sigma = 1\}$.

Agents' problem is reformulated to fit event-tree $\mathcal{D}^T(\xi_0)$. In particular, agent $h \in \mathcal{H}$ is characterized by a modified utility functional over consumption streams, $U^{h,T} = \sum_{\xi \in \mathcal{D}^T(\xi_0)} u^h(\xi, x(\xi))$, and commodity and financial endowments $(w^h(\xi), e^h(\xi))_{\xi \in \mathcal{D}^T(\xi_0)} \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{D}^T(J)}$. More precisely, we assume that each h receives a financial endowment of $\frac{1}{n} > 0$ of asset k at node ξ_k, for every asset $k \in \mathcal{J}^T$ available in economy $\mathcal{E}_n^T$:

$$e_k^h(\xi) = \begin{cases} \frac{1}{n} & \xi = \xi_k. \\ 0 & \xi \neq \xi_k. \end{cases}$$

Let $\bar{e}_j^h(\xi)$ stand for the cumulative endowment of asset j received by agent h up to node ξ. The deliveries of assets in $\mathcal{J}(\xi^-)$, scaled by $\frac{1}{n}$, should now be considered as part of each agent's endowments, and thus, accounted for in aggregate wealth at ξ:

$$W_n(\xi) = W(\xi) + \sum_{h \in \mathcal{H}} \left(\frac{1}{n} \sum_{k \in \mathcal{J}^T(\xi^-)} A_k(\xi) \right).$$

Let $W_n^T = (W_n(\xi))_{\xi \in \mathcal{D}^T(\xi_0)}$ stand for economy $\mathcal{E}_n^T$'s aggregate wealth throughout event-tree $\mathcal{D}^T(\xi_0)$.

Each agent must choose an allocation $(y^h(\xi))_{\xi \in \mathcal{D}^T(\xi_0)} = (x^h(\xi), \theta^h(\xi), \varphi^h(\xi))_{\xi \in \mathcal{D}^T(\xi_0)}$ belonging to space $\mathbb{E}^T := \mathbb{R}_+^{\mathcal{D}^T(\xi_0) \times \mathcal{L}} \times \mathbb{R}_+^{\mathcal{D}^T(\mathcal{J})} \times \mathbb{R}_+^{\mathcal{D}^T(\mathcal{J})}$. We denote $y^h = (y^h(\xi))_{\xi \in \mathcal{D}^T(\xi_0)}$. For prices $(p, q) \in \mathbb{P}^T$, agent h's truncated choice set correspondence $\mathcal{C}^{h,T}(p, q, \kappa)$ considers allocations $y^h \in \mathbb{E}^T$ complying with constraints:

$$g^{h,T}(\xi, y^h(\xi), y^h(\xi^-); p, q) \leq 0, \quad \forall \xi \in \mathcal{D}^T(\xi_0),$$
$$q(\xi)\varphi^h(\xi) \leq \kappa p(\xi) W(\xi),$$

where $(\theta^h(\xi_0^-), \varphi^h(\xi_0^-)) = (0,0)$ and for every $\xi \in \mathcal{D}^T(\xi_0)$ the function $g^{h,T}(\xi, .)$ is given by:

$$\begin{aligned} g^{h,T}(\xi, y(\xi), y(\xi^-); p, q) \coloneqq\ & p(\xi)\left(x^h(\xi) - w^h(\xi)\right) + q(\xi)(\theta^h(\xi) - \varphi^h(\xi) - e^h(\xi)) \\ & - \sum_{j \in \mathcal{J}^T(\xi^-)} (p(\xi)A_j(\xi) + q_j(\xi))(\theta_j^h(\xi^-) - \varphi_j^h(\xi^-)). \end{aligned}$$

Definition 2. A *competitive equilibrium* for economy $\mathcal{E}_n^T$ is composed by a price process $(p,q) \in \mathbb{P}^T$ and allocations $(y^h)_{h\in\mathcal{H}} \in (\mathbb{E}^T)^{\mathcal{H}}$ such that

(1) For every $h \in \mathcal{H}$, $y^h \in \operatorname{argmax}_{y \in C^{h,T}(p,q)} U^{h,T}(x)$.

(2) Physical and financial markets clear, i.e.,

$$\sum_{h\in\mathcal{H}} x^h(\xi) = W_n(\xi) \quad \forall \xi \in \mathcal{D}^T(\xi_0), \quad \text{and} \quad \sum_{h\in\mathcal{H}} \theta^h(\xi) = \sum_{h\in\mathcal{H}} (\varphi^h(\xi) + \bar{e}^h(\xi)) \ \forall \xi \in \mathcal{D}^{T-1}(\xi_0).$$

Finite horizon economy $\mathcal{E}^T$ is defined equivalently to $\mathcal{E}_n^T$, with the exception that there are no perturbations in the net supply of financial securities: $e^h(\xi) = 0$, for all $(h,\xi) \in \mathcal{H} \times \mathcal{D}^{T-1}(\xi_0)$. Analogously, infinite horizon economy $\mathcal{E}_n$ is equivalent to $\mathcal{E}$ with the modified net supply of financial assets.

Equilibrium in truncated economies. Assumptions B1-B4 ensure that, for every agent $h \in \mathcal{H}$, $U^h(W_n^T) < +\infty$. Thus, and as all assets are in positive net supply, equilibrium existence for economy $\mathcal{E}_n^T$ follows directly from Lemma 2 of Moreno-García and Torres-Martínez (2012). Furthermore, the truncation of assets' net supply implies that at every node there are agents maintaining investments in each of the available securities.

Now, consider any competitive equilibrium $\left[(\bar{p}_n, \bar{q}_n); (\bar{y}_n^h)_{h\in\mathcal{H}}\right]$ of $\mathcal{E}_n^T$. Assumption B1 allows us to assert that at every $\xi \in \mathcal{D}^T(\xi_0)$ and for any $n \in \mathbb{N}$ there exists a bundle of commodities $a_n(\xi) \in \mathbb{R}_{++}^{\mathcal{L}}$ providing more utility than the one attainable trough aggregate wealth in the infinite horizon economy $\mathcal{E}_n$:

$$u^h(\xi, a_n(\xi)) > U^h(W_n), \ \forall h \in \mathcal{H}.$$

Analogously, Assumptions B1 and B2 ensure that we may define bundles $(a(\xi))_{\xi\in\mathcal{D}^{T-1}(\xi_0)} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{D}^{T-1}(\xi_0)}$ such that:

$$u^h(\xi, a(\xi)) > U^h(W), \ \forall h \in \mathcal{H},$$

where W stands for aggregate wealth at original economy $\mathcal{E}$. Furthermore, Assumption B3 plus the continuity of utility functions allows us to assert that there exists $\overline{N} \in \mathbb{N}$ such that:

$$u^h(\xi, a(\xi)) > U^h(W_n), \ \forall h \in \mathcal{H},$$

and for any $n \geq \overline{N}$. We assume this holds from now on.

Because equilibrium allocations are bounded by aggregate wealth, results by Moreno García and Torres-Martínez (2012) allow us to assert that asset prices at any $\xi \in \mathcal{D}^{T-1}(\xi_0)$ have upper bounds which are dependent on their net supply:

$$\bar{q}_{n,k}(\xi) \leq n\|a(\xi)\| \text{ for all } k \in \mathcal{J}^T(\xi).$$

We now proceed to show that Assumptions B1-B3, jointly with a sufficiently large value of κ, induce upper bounds on prices of securities in $\mathcal{K}^T$, which are independent of both the net supply of securities and the time horizon of the truncated economy.

Lemma 6.1. *There exists $M = (M(\xi))_{\xi \in D}$, independent of $n \in \mathbb{N}$, such that the security prices $\bar{q}_{n,j}(\xi) \leq M(\xi)$ for all $(\xi, j) \in \mathcal{D}^T(\xi_0) \times \mathcal{K}^T(\xi)$, given that κ is large enough.*

Proof. For each $(j, \xi) \in \mathcal{K}^T(\xi) \times \mathcal{D}^T(\xi_0)$ such that $\xi = \xi_j$, Assumption B3 ensures there exists $\delta > 0$ such that some agent $h \in \mathcal{H}$ may short sell δ units of j at ξ and honor her commitments with her future endowment streams.

Assume that $\kappa \geq \hat{\kappa}(\xi) = \frac{\|a(\xi)\|}{\min\{\underline{w}(\xi),1\}}$, where $\underline{w}(\xi) = \min_{(h,l)\in\mathcal{H}\times\mathcal{L}}\{w_l^h(\xi)\}$, and is correctly determined due to Assumption B2. I claim that this implies that the price of j at ξ is bounded:

$$\bar{q}_{j,n}(\xi) \leq M_j(\xi) := \frac{\|a(\xi)\|}{\hat{\kappa}(\xi) \times \delta}.$$

Otherwise, h could short-sell δ units of j, use those resources to purchase and consume bundle $a(\xi)$, and pay her debt with her future endowment streams. That is, she could implement the following allocation $(\tilde{x}, \tilde{\theta}, \tilde{\varphi}) \in \mathbb{E}^T$:

$$\tilde{\varphi}_k(\eta) = \begin{cases} -\delta & \text{for } \eta \geq \xi \text{ and } k = j, \\ 0 & \text{otherwise.} \end{cases} \quad \tilde{x}(\eta) = \begin{cases} a(\eta) & \text{for } \eta = \xi, \\ w^h(\eta) - \delta A_j(\eta) & \text{for } \eta > \xi, \\ w^h(\eta) & \text{otherwise.} \end{cases} \quad \tilde{\theta}(\eta) = 0,\ \forall \eta.$$

By doing so, she would obtain a greater utility than the one achievable in economy $\mathcal{E}_n$, which constitutes a contradiction. Furthermore, the interiority of endowments ensures that all agents have at ξ a minimum of $\underline{w}(\xi)$ resources to spend, and thus, all agents may purchase at least $\varepsilon_j = \frac{M_j(\xi)}{\underline{w}(\xi)} > 0$ units of asset j at ξ. Again, this implies that for every $\mu > \xi$ the price of j must be bounded:

$$\bar{q}_{j,n}(\mu) \leq M_j(\mu) := \frac{\|a(\mu)\|}{\varepsilon_j}.$$

Otherwise, agents could implement an allocation $(\check{x}, \check{\theta}, \check{\varphi}) \in \mathbb{E}^T$ that involves purchasing ε_j units of j at ξ and selling her holdings at μ as to purchase $a(\mu)$:

$$\check{\theta}_k(\eta) = \begin{cases} \varepsilon_j & \text{for } \mu > \eta \geq \xi,\ k = j, \\ 0 & \text{otherwise.} \end{cases} \quad \check{x}(\eta) = \begin{cases} 0 & \text{for } \eta = \xi, \\ a(\eta) & \text{for } \eta = \mu, \\ w^h(\eta) & \text{otherwise.} \end{cases} \quad \check{\varphi}(\eta) = 0,\ \forall \eta.$$

Again, this leads to a contradiction as allocation $(\check{x}, \check{\theta}, \check{\varphi})$ provides a greater utility than the one feasible in $\mathcal{E}_n$.

Let $\mathcal{D}^T(\mathcal{K}^T) = \{\xi \in \mathcal{D}^T(\xi_0) : \xi = \xi_j \text{ for } j \in \mathcal{K}^T\}$, and define $\hat{\kappa}^T = \max_{\xi \in \mathcal{D}^T_{\mathcal{K}}(\xi_0)}\{\hat{\kappa}(\xi)\}$ as well as $M(\xi) = \max_{j \in J^T(\xi)}\{M_j(\xi)\}$. Because the election of $j \in \mathcal{K}^T$ was arbitrary, we learn that prices of securities in $\mathcal{K}^T$ are bounded by $M = (M(\xi))_{\xi \in \mathcal{D}^T(\xi_0)}$ in any competitive equilibrium of $\mathcal{E}^T_n$ in which $\kappa \geq \hat{\kappa}^T$. This ends the proof. □

Because the cardinality of $\mathcal{K}$ is finite (Assumption B3), it holds that $\mathcal{K}^T = \mathcal{K}$ provided that $T \in \mathbb{N}$ is sufficiently large. Hence, for T large enough, the set $\mathcal{D}^T(\mathcal{K}^T)$ coincides with $D(\mathcal{K}) = \{\xi \in D : \xi = \xi_j \text{ for } j \in \mathcal{K}\}$, and is comprised by a finite number of nodes. If follows that there exists a fixed value $\hat{\kappa}$ for which Lemma 6.1 holds for any economy $\mathcal{E}^T_n$. We assume that $\kappa > \hat{\kappa}$ from now on.

It is important to highlight that bounds M are independent from the perturbation on the net supply of financial securities, provided $n > \overline{N}$. Moreover, at ξ bounds $M(\xi)$ are independent of the truncation horizon, provided that $\mathcal{K}^T(\xi) = \mathcal{K}(\xi)$ (which holds for T large enough). We now proceed to show that Lemma 6.1, jointly with Assumptions B1 and B4 imply there exist bounds on prices of securities in $\mathcal{J}^T \setminus \mathcal{K}^T$, which are also independent of both the net supply of securities and the time horizon of the truncated economy.

Lemma 6.2. *There exists $N = (N(\xi))_{\xi \in D}$, independent of $n \in \mathbb{N}$, such that the security prices $\bar{q}_{n,k}(\xi) \leq N(\xi)$ for all $(\xi, j) \in \mathcal{D}^T(\xi_0) \times \mathcal{J}^T(\xi) \setminus \mathcal{K}^T(\xi)$.*

Proof. Recall that Assumption B4 ensures that for every security in $\mathcal{J}^T \setminus \mathcal{K}^T$ there is a portfolio $\hat{\theta}(j)$ comprised solely of securities in $\mathcal{K}^T$ which always provides a greater delivery stream. Furthermore, note that because securities in $\mathcal{J}^T$ are in positive net supply, in equilibrium there is always some agent who purchases a positive amount of those securities.

Now, consider a pair $(\xi, j) \in \mathcal{D}^T(\xi_0) \times \mathcal{J}^T(\xi) \setminus \mathcal{K}^T(\xi)$ such that $t(\xi) = T - 1$. We claim that:

$$\bar{q}_{n,j}(\xi) \leq \sum_{j \in \mathcal{K}^T(\xi)} (\bar{q}_{n,j}(\xi) \times \hat{\theta}_k(j)). \tag{3}$$

Otherwise, any agent investing in k would be better off by using those resources to purchase portfolio $\hat{\theta}(j)$ instead. Indeed, by pursuing this strategy she would be increasing her consumption at ξ while maintaining the rest of her consumption plan unaltered. This would constitute a contradiction with the optimality of h's behavior. As the election of (ξ, j) was arbitrary, we learn that (3) holds for every security $j \in \mathcal{J}^T(\xi) \setminus \mathcal{K}^T(\xi)$, and for any node ξ such that $t(\xi) = T - 1$.

Note that we may sequentially apply the same argument for securities available in nodes at dates $T - 2$, $T - 3$, and so on. That is, and because agents optimally decide to purchase them, the price of securities in $\mathcal{J}^T \setminus \mathcal{K}^T$ never exceeds the price of their corresponding super-replicating portfolio.

Moreover, from Lemma 6.1 we know that the prices of securities in $\mathcal{K}^T$ are bounded by M. Therefore, the price of securities in $\mathcal{J}^T \setminus \mathcal{K}^T$ are bounded by:

$$\bar{q}_{n,j}(\xi) \leq N(\xi) := \max_{j \in \mathcal{J}^T(\xi) \setminus \mathcal{K}^T(\xi)} \left\{ M(\xi) \times \sum_{k \in \mathcal{K}^T(\xi)} \hat{\theta}_k(j) \right\}.$$

Defining $N = (N(\xi))_{\xi \in \mathcal{D}^T(\xi_0)}$ ends the proof. □

Because bounds M do not depend on the truncation of the positive net supply (given that $n \geq \bar{N}$), it holds that bounds N do not depend on n as well. Moreover, at any node ξ, bounds $N(\xi)$ are independent from the truncation horizon provided that $\mathcal{J}^T(\xi) = \mathcal{J}(\xi)$ (which holds for T large enough).

Consider the sequence consisting of equilibrium prices and allocations given by

$$\left[(\bar{p}_n, \bar{q}_n); (\bar{y}^h_n)_{h\in\mathcal{H}}\right]_{n\geq\overline{N}}.$$

The results exposed by Moreno-García and Torres-Martínez (2012, Lemma 2) ensure that the latter is node-by-node bounded. Hence, we may use Tychonoff's theorem to ensure it has a convergent subsequence $(n_k)_{k\in\mathbb{N}}$ such that:

$$\lim_{n_k\to+\infty}\left[(\bar{p}_{n_k}, \bar{q}_{n_k}); (\bar{y}^h_{n_k})_{h\in\mathcal{H}}\right] = \left[(\bar{p}, \bar{q}); (\bar{y}^h)_{h\in\mathcal{H}}\right].$$

We treat $\left[(\bar{p}, \bar{q}); (\bar{y}^h)_{h\in\mathcal{H}}\right]$ as our candidate for a competitive equilibrium for economy $\mathcal{E}^T$. As equilibrium allocations comply with market clearing and belong to agent's budget sets for every $n \in \mathbb{N}$, the same holds for the limit allocations. Thus, we must solely show that the limit allocation is optimal for agents regarding the limit's prices.

By contradiction, assume that for some agent h there exists $\tilde{y} = (\tilde{x}, \tilde{\theta}, \tilde{\varphi}) \in \mathcal{C}^{h,T}(\bar{p}, \bar{q}, \kappa)$ such that $U^{h,T}(\tilde{x}) > U^{h,T}(\bar{x})$. The continuity of $\mathcal{C}^{h,T}(.)$ allows to assert that there exists $\{\tilde{y}_{n_k}\}_{k\in\mathbb{N}}$ complying with $\lim_{k\to+\infty}\tilde{y}_{n_k} = \tilde{y}$ and $\tilde{y}_{n_k} \in \mathcal{C}^{h,T}(\bar{p}, \bar{q}, \kappa)$ for all n_k. The continuity of $U^{h,T}$ then implies that there exists n^*_k large enough such that $U^{h,T}(\tilde{x}_{n_k}) > U^{h,T}(\bar{x}_{n_k})$ for all $n_k \geq n^*_k$. This leads to a contradiction, as we already know $\bar{x}_{n_k}$ is optimal regarding h's budget set. Thus, we have proved the existence of a competitive equilibrium for economy $\mathcal{E}^T$.

As the election of $T \in \mathbb{N}$ was arbitrary, we know a competitive equilibrium exists for any finite horizon truncated economy, $\mathcal{E}^T$. Equilibrium existence for the infinite horizon case then follows directly from Moreno-García and Torres-Martínez (2012, *Asymptotic equilibria*, p.141).

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